

A REPRESENTATION THEOREM FOR MEASURABLE RELATION ALGEBRAS WITH CYCLIC GROUPS

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ABSTRACT. A relation algebra is measurable if the identity element is a sum of atoms, and the square $x;1;x$ of each subidentity atom x is a sum of non-zero functional elements. These functional elements form a group G_x . We prove that a measurable relation algebra in which the groups G_x are all finite and cyclic is completely representable. A structural description of these algebras is also given.

1. INTRODUCTION

Relation algebras were defined by Tarski as a class of abstract algebras satisfying ten equations, to serve as an algebraic counterpart of logic. They are Boolean algebras with operators where the extra-Boolean operators $;$, \smile , $1'$ form an involuted monoid (satisfying some further equations). In particular, the complex algebra of a group is a relation algebra. Algebras of binary relations with standard set theoretic Boolean operations and relation-composition, inverse of a relation and set theoretic identity relation as the extra-Boolean operations are called *set relation algebras*. These are relation algebras, and a relation algebra isomorphic to a set relation algebra is called *representable*. In particular, the function assigning the Cayley-representation to each element of a group extends to a representation of the complex algebra of the group.

Measurable relation algebras were introduced in [6]. In a relation algebra, a subidentity atom x is defined to be measurable if $x;1;x$ is the sum of functional elements and a relation algebra is *measurable* if $1'$ is the sum of measurable atoms. (Atoms below the identity $1'$ are called subidentity atoms, and an element f is called functional if $f\smile; f \leq 1'$.) Measurable relation algebras are closely related to groups. They are put together from various groups as follows. The non-zero functional elements below $x;1;x$ are atoms and they form a group G_x with identity element x under the operations of $;$ and \smile , and the atoms below $x;1;y$ specify isomorphisms between quotient groups G_x/H_{xy} and G_y/K_{xy} , when x, y are distinct subidentity atoms (see [6, p.53], [3] and Section 4 in this paper).

A group pair is defined as a system of groups G_x with normal subgroups H_{xy} and K_{xy} , and isomorphisms φ_{xy} between quotient groups G_x/H_{xy} and G_y/K_{xy} , for $x, y \in I$. When the isomorphisms are linked by certain natural conditions, we can put together the Cayley-representations of the various groups occurring in the group pair \mathcal{F} to get a set relation algebra $\mathfrak{G}[\mathcal{F}]$. A *group frame* is a group pair that satisfies the natural conditions, given in this paper as Definition 3.7. The

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algebras $\mathfrak{G}[\mathcal{F}]$ constructed from group frames are called (generalized full) *group relation algebras*. Group relation algebras are all measurable, atomic, complete set algebras, with all suprema being union in them (i.e., completely representable). For examples and illustrations see [6] and [3].

An immediate question is whether group relation algebras exhaust the examples of all atomic complete measurable relation algebras.

This paper proves that indeed this is the case when all the groups G_x in the measurable relation algebra are finite and cyclic (Representation Theorem 5.7). Even the conditions of being atomic and complete can be omitted: all measurable relation algebras \mathfrak{A} with finite cyclic groups are atomic and *essentially* isomorphic to group relation algebras. This latter means that the completion (the minimal complete extension) of \mathfrak{A} is isomorphic to a group relation algebra. The passage to the completion of \mathfrak{A} does not change the structure of \mathfrak{A} , it only fills in any missing infinite sums that are needed in order to obtain isomorphism with the necessarily complete full group relation algebra. We note that an atomic measurable relation algebra is essentially isomorphic to a group relation algebra if and only if it is completely representable ([7, Theorems 7.4, 7.6]).

In the case when all the groups are cyclic, the group frame conditions considerably simplify, see Definition 4.1 here. This fact gives measurable relation algebras with finite cyclic groups an especially clear structural description.

The algebras that come up in Theorem 4.30 of Jónsson-Tarski [9] are all measurable with associated groups being one-element. Hence, the hard direction (ii) \Rightarrow (i) of [9, Theorem 4.30] follows from our Theorem 5.7. Algebras with all the associated groups being cyclic of order one or two also have come up in the literature. We show, in Lemma 5.8, that among atomic relation algebras they are exactly the pair-dense algebras of Maddux [10]. This gives a possibility for giving a new proof for [10, Theorem 51].

An extension of Representation Theorem 5.7 is known. If \mathfrak{A} is a measurable relation algebra in which the group G_x is a product of two finite cyclic groups for each subidentity atom x , then \mathfrak{A} is essentially isomorphic to a group relation algebra (this is announced in [6, Theorem 6]). The proof of this extended theorem (due to the authors) is complicated, and will not be given here.

The theorem cannot be extended to the case in which the groups G_x are allowed to be products of three finite cyclic groups. Indeed, an example is presented in [6, pp.56-59], and proved to be nonrepresentable in [1, Theorem 5.2], of a finite measurable relation algebra with five measurable atoms x such that, the group G_x is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, for all x .

There are several other instances in which an atomic measurable relation algebra is essentially isomorphic to a group relation algebra. This proves to be the case, for example, if there are at most four measurable atoms in all. It is also the case if, for every measurable atom x , every chain of normal subgroups of G_x has length at most three (so that there cannot be normal subgroups H and K of G_x such that $\{x\} \subsetneq H \subsetneq K \subsetneq G_x$). Another case is when the normal subgroups K_{xy} and H_{yz} are always the same. See [6, pp.56-59].

As an application of some of these ideas, let us paraphrase Maddux's terminology by calling a relation algebra *n-dense* if it is measurable, and if, for each measurable atom x , the group G_x has cardinality at most n . Every group of cardinality at most 7 is either cyclic, the product of two cyclic groups, or has no normal subgroup chains

of length more than three. As a result, we obtain that every 7-dense relation algebra is representable, but there exist 8-dense relation algebras (with five measurable atoms) that are not representable.

The above results are summarized in [6] (without proofs), along with detailed motivation and many illustrations. Readers who wish to learn more about the subject of relation algebras are recommended to look at the books by Hirsch-Hodkinson [8], Maddux [11], or Givant [4], [5].

The remainder of the paper is divided into four sections. Section 2 reviews the necessary relation algebraic background for reading the paper. Throughout the paper, there are references to the earlier papers [3], [1], and [7]; the results needed from those papers are explained as they are encountered. In Section 3, a characterization is given, in terms of a system of simple invariants, of when it is possible to construct a full group relation algebra from a given system of mutually disjoint, finite, cyclic groups. Section 4 discusses the notion of a regular element—a kind of generalization of the notion of an atom—and of the index of such an element. Some important properties of indices are formulated and proved. Finally, Section 5 is devoted to a proof of the main theorem of the paper, namely to Representation Theorem 5.7.

2. RELATION ALGEBRAS

In the next few sections, most of the calculations will involve the arithmetic of relation algebras. This section provides a review of the essential results that will be needed.

A relation algebra is an algebra of the form

$$\mathfrak{A} = (A, +, -, ;, \smile, 1'),$$

where $+$ and $;$ are binary operations called *addition* and *relative multiplication*, while $-$ and \smile are unary operations called *complement* and *converse*, and $1'$ is a distinguished constant called the *identity element*, such that ten equational axioms hold in \mathfrak{A} . The exact nature of these axioms is not important for the present discussion. Further operations and relations such as Boolean multiplication \cdot and the Boolean partial order relation \leq are defined in the standard way. The following laws, which are either members of Tarski's ten axioms or are derivable from them, play a role in this paper. For their derivation, see e.g., [4].

Lemma 2.1. *If $\mathfrak{A} = (A, +, -, ;, \smile, 1')$ is a relation algebra, then $(A, +, -)$ is a Boolean algebra, and the operation of converse is an automorphism of this Boolean algebra. In particular, the following laws hold.*

- (i) $(a + b)^\smile = a^\smile + b^\smile$.
- (ii) $(a \cdot b)^\smile = a^\smile \cdot b^\smile$.
- (iii) $a \leq b$ if and only if $a^\smile \leq b^\smile$.
- (iv) a is an atom if and only if a^\smile is an atom.
- (v) $x^\smile = x$ and $x = x$ whenever x is a subidentity element.

Properties (i) and (iii) are called the *distributive* and *monotony laws* for converse.

Lemma 2.2.

- (i) $r; (s; t) = (r; s); t$.
- (ii) $r; 1' = r$.
- (iii) $r^{\smile\smile} = r$.

- (iv) $(r; s)^\smile = s^\smile; r^\smile$.
- (v) $(r + s); t = r; t + s; t$.
- (vi) If $a \leq b$ and $c \leq d$, then $a; c \leq b; d$.

These properties are commonly referred to by the following names: the *associative law for relative multiplication*, the (right-hand) *identity law for relative multiplication*, the *first involution law*, the *second involution law*, the (right-hand) *distributive law for relative multiplication*, and the *monotony law for relative multiplication*. An element x in \mathfrak{A} is called a *subidentity element* if it is below the identity element, in symbols $x \leq 1'$. Whenever parentheses indicating the order of performing operations are lacking, it is understood that unary operations have priority over binary operations, and multiplications have priority over addition.

A *square* is an element of the form $x; 1; x$ for some subidentity element x , and a *rectangle* is an element of the form $x; 1; y$ for some subidentity elements x and y . The elements x and y are sometimes referred to as the *sides* of the rectangle.

Lemma 2.3. *Let x, y, z, w be subidentity elements.*

- (i) $(x; 1; y)^\smile = y; 1; x$.
- (ii) $(x; 1; y); b \leq x; 1; z$ for every $b \leq y; 1; z$, and equality holds whenever x, y , and z are atoms, and $x; 1; y$ and b are both non-zero.
- (iii) If x and y are subidentity atoms, and if $0 \leq b \leq x; 1; y$, then $x; b = b; y = b$.

3. GROUP RELATION ALGEBRAS WITH FINITE CYCLIC GROUPS

Let $G = \langle G_x : x \in I \rangle$ be a system of pairwise disjoint, finite, cyclic groups $(G_x, \circ, ^{-1}, e_x)$, and

$$m = \langle m_{xy} : (x, y) \in \mathcal{E} \rangle,$$

a system of positive integers indexed by an equivalence relation \mathcal{E} on the index set I . We may assume that for each x in I , the cyclic group G_x is a copy of the cyclic group \mathbb{Z}_{n_x} of integers modulo n_x for some positive integer n_x . We shall usually act and write as if the two groups were identical, although technically it is important to pass to a copy of \mathbb{Z}_{n_x} in order to achieve the assumed disjointness of the groups in the system G . The greatest common divisor of two numbers m, n is denoted by $\gcd(m, n)$. For the following definition see also [6, p.51].

Definition 3.1. The system of indices

$$m = \langle m_{xy} : (x, y) \in \mathcal{E} \rangle,$$

is said to satisfy the *index conditions* if the following conditions hold.

- (i) m_{xy} is a common divisor of the orders of G_x and G_y .
- (ii) m_{xx} is equal to the order of G_x .
- (iii) $m_{yx} = m_{xy}$.
- (iv) $\gcd(m_{xy}, m_{yz}) = \gcd(m_{xy}, m_{xz}) = \gcd(m_{xz}, m_{yz})$.

Assume m does satisfy the index conditions, and write

$$d = \gcd(m_{xy}, m_{xz}) = \gcd(m_{xy}, m_{yz}).$$

This equation holds because of index condition (iv). For each pair (x, y) in \mathcal{E} , take H_{xy} and K_{xy} be the respective subgroups of G_x and G_y that consist of the multiples of m_{xy} . This definition makes sense because of the first index condition. It follows

that H_{xy} and K_{xy} have index m_{xy} in G_x and G_y respectively, that is to say, they each have m_{xy} cosets, in symbols,

$$m_{xy} = |G_x/H_{xy}| = |G_x/K_{xy}|.$$

Notice that these subgroups are always normal, since the groups in the system G are all cyclic, and hence Abelian. Because m_{xy} is the index in G_x of the subgroup H_{xy} , this subgroup must consist of the multiples of the integer m_{xy} modulo n_x . In particular, the cosets of H_{xy} are the sets of the form

$$H_{xy} + \ell = \{pm_{xy} + \ell : p < n_x/m_{xy}\}$$

for $\ell < m_{xy}$.

The composite group

$$H_{xy} \circ H_{xz} = \{h \circ k : h \in H_{xy} \text{ and } k \in H_{xz}\}$$

consists of the multiples of d modulo n_x , and the cosets of $H_{xy} \circ H_{xz}$ are the sets of the form

$$H_{xy} \circ H_{xz} + s$$

for $0 \leq s < d$.

Lemma 3.2. *For each integer s with $0 \leq s < d$,*

$$\begin{aligned} H_{xy} \circ H_{xz} + s &= \bigcup \{H_{xy} + \ell : 0 \leq \ell < m_{xy} \text{ and } \ell \equiv s \pmod{d}\} \\ &= \bigcup \{H_{xy} + qd + s : q < m_{xy}/d\}. \end{aligned}$$

Proof. Each non-negative integer $\ell < m_{xy}$ can be written in one and only one way in the form

$$\ell = qd + s$$

for some integers s and q satisfying $0 \leq s < d$ and $0 \leq q < m_{xy}/d$, by the division algorithm for integers, so the second equality of the lemma is clear.

Observe that H_{xy} is included in the composite group $H_{xy} \circ H_{xz}$. Also, d generates the composite group, so qd is in the composite group for every integer q with $0 \leq q < m_{xy}/d$. Combine these observations to see that

$$H_{xy} + qd \subseteq H_{xy} \circ H_{xz},$$

and therefore

$$H_{xy} + qd + s \subseteq H_{xy} \circ H_{xz} + s,$$

for every $s < d$ and every $q < m_{xy}/d$. It follows that

$$(1) \quad \bigcup \{H_{xy} + qd + s : q < m_{xy}/d\} \subseteq H_{xy} \circ H_{xz} + s,$$

for every $s < d$.

The cosets that make up the union on the left side of (1) partition G_x as q and s vary, since there are assumed to be m_{xy} such cosets, one for each $\ell = qd + s < m_{xy}$. Also the cosets on the right partition G_x as s varies. It follows that equality must hold in (1). In more detail, if f belongs to the right side of (1), then $f \equiv s \pmod{d}$. Consequently, f cannot belong to any of the cosets $H_{xy} + qd + t$ for $0 \leq t < d$ and $t \neq s$, since the elements in these cosets are congruent to t modulo d . Thus, f must belong to $H_{xy} + qd + s$ for some q with $0 \leq q < m_{xy}/d$. \square

One sees in a similar fashion that the subgroup K_{xy} has n_y/m_{xy} elements and m_{xy} cosets, which have the form $K_{xy} + \ell$ for $\ell < m_{xy}$. The composite subgroup $K_{xy} \circ H_{yz}$ is generated by d , and has n_y/d elements and d cosets

$$K_{xy} \circ H_{yz} + s$$

for $0 \leq s < d$. The proof of the next lemma is very similar to that of Lemma 3.2, and will therefore be omitted.

Lemma 3.3. *For each integer s with $0 \leq s < d$,*

$$\begin{aligned} K_{xy} \circ H_{yz} + s &= \bigcup \{K_{xy} + \ell : 0 \leq \ell < m_{xy} \text{ and } \ell \equiv s \pmod{d}\} \\ &= \bigcup \{K_{xy} + qd + s : q < m_{xy}/d\}. \end{aligned}$$

Define a mapping φ_{xy} from G_x/H_{xy} to G_y/K_{xy} by

$$\varphi_{xy}(H_{xy} + \ell) = K_{xy} + \ell$$

for $0 \leq \ell < m_{xy}$. The mapping is certainly a bijection, by the preceding remarks, and it maps the generator $H_{xy} + 1$ of the quotient group G_x/H_{xy} to the generator $K_{xy} + 1$ of the quotient group G_y/K_{xy} , so it must be an isomorphism, as is easy to check directly. This isomorphism induces an isomorphism $\hat{\varphi}_{xy}$ from $G_x/(H_{xy} \circ H_{xz})$ to $G_y/(K_{xy} \circ H_{yz})$.

Lemma 3.4. $\hat{\varphi}_{xy}(H_{xy} \circ H_{xz} + s) = K_{xy} \circ H_{yz} + s$ for $0 \leq s < d$.

Proof. Use the definition of $\hat{\varphi}_{xy}$, Lemma 3.2, the definition of φ_{xy} , and Lemma 3.3 to obtain

$$\begin{aligned} \hat{\varphi}_{xy}(H_{xy} \circ H_{xz} + s) &= \varphi_{xy}[\bigcup \{H_{xy} + qd + s : 0 \leq q < m_{xy}/d\}] \\ &= \bigcup \{\varphi_{xy}(H_{xy} + qd + s) : 0 \leq q < m_{xy}/d\} \\ &= \bigcup \{K_{xy} + qd + s : 0 \leq q < m_{xy}/d\} \\ &= K_{xy} \circ H_{yz} + s. \end{aligned}$$

□

In a similar fashion, there is a quotient isomorphism φ_{yz} from G_y/H_{yz} to G_z/K_{yz} that is defined by

$$\varphi_{yz}(H_{yz} + \ell) = K_{yz} + \ell$$

for $0 \leq \ell < m_{yz}$. This isomorphism, in turn, induces an isomorphism $\hat{\varphi}_{yz}$ from $G_y/(K_{xy} \circ H_{yz})$ to $G_z/(K_{xz} \circ K_{yz})$ that satisfies the following lemma.

Lemma 3.5. $\hat{\varphi}_{yz}(K_{xy} \circ H_{yz} + s) = K_{xz} \circ K_{yz} + s$ for $0 \leq s < d$.

Finally, there is a quotient isomorphism φ_{xz} from G_x/H_{xz} to G_z/K_{xz} that is defined by

$$\varphi_{xz}(H_{xz} + \ell) = K_{xz} + \ell$$

for $0 \leq \ell < m_{xz}$. This isomorphism induces an isomorphism $\hat{\varphi}_{xz}$ from $G_x/(H_{xy} \circ H_{xz})$ to $G_z/(K_{xz} \circ K_{yz})$ that satisfies the following lemma.

Lemma 3.6. $\hat{\varphi}_{xz}(H_{xy} \circ H_{xz} + s) = K_{xz} \circ K_{yz} + s$ for $0 \leq s < d$.

The following definition is from [6, Definition 1], see also [3, Definition 4.1]).

Definition 3.7. A *group frame* is a group pair

$$\mathcal{F} = (\langle G_x : x \in I \rangle, \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle)$$

satisfying the following *frame conditions* for all pairs (x, y) and (y, z) in \mathcal{E} .

- (i) φ_{xx} is the identity automorphism of $G_x/\{e_x\}$ for all x .
- (ii) $\varphi_{yx} = \varphi_{xy}^{-1}$.
- (iii) $\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}$.
- (iv) $\hat{\varphi}_{xy}|\hat{\varphi}_{yz} = \hat{\varphi}_{xz}$.

□

It is shown in [3] that this definition gives necessary and sufficient conditions for a group pair \mathcal{F} to give rise to a group relation algebra $\mathfrak{G}[\mathcal{F}]$. We recall the definition of $\mathfrak{G}[\mathcal{F}]$ from [6], [3]. Suppose that $(H_{xy,\alpha} : \alpha < \kappa_{xy})$ is a listing of the cosets of H_{xy} in G_x . Define

$$R_{xy,\alpha} = \bigcup_{\gamma < \kappa_{xy}} H_{xy,\gamma} \times \varphi_{xy}(H_{xy,\gamma} \circ H_{xy,\alpha}).$$

Let A be the set of all binary relations of form $\bigcup\{R_{xy,\alpha} : (x,y,\alpha) \in X\}$, where $X \subseteq \{(x,y,\alpha) : (x,y) \in \mathcal{E} \text{ and } \alpha < \kappa_{xy}\}$. When \mathcal{F} is a group frame, then A is a set of binary relations that is closed under the Boolean set-theoretic operations, that contains the identity relation on $\bigcup\{G_x : x \in I\}$, and that is closed under the operations of forming the composition of two binary relations and the converse of a binary relation. The set relation algebra with universe A is denoted by $\mathfrak{G}[\mathcal{F}]$. It is easy to see that each supremum in $\mathfrak{G}[\mathcal{F}]$ is indeed a union, so $\mathfrak{G}[\mathcal{F}]$ is completely represented.

Theorem 3.8 (GCD Theorem). *Let $G = \langle G_x : x \in I \rangle$ be a system of mutually disjoint, finite, cyclic groups, and \mathcal{E} an equivalence relation on I . For each system*

$$m = \langle m_{xy} : (x,y) \in \mathcal{E} \rangle$$

of positive integers satisfying the index conditions, there exists a system of quotient isomorphisms $\varphi = \langle \varphi_{xy} : (x,y) \in \mathcal{E} \rangle$ such that the group pair $\mathcal{F} = (G, \varphi)$ satisfies the four frame conditions and is therefore a group frame. The corresponding group relation algebra $\mathfrak{G}[\mathcal{F}]$ therefore exists. Moreover,

$$m_{xy} = |G_x/H_{xy}|,$$

where H_{xy} is the kernel of φ_{xy} .

Proof. Consider, first, frame condition (i). Index condition (ii) implies that m_{xx} coincides with the cardinality n_x of the group. The subgroups H_{xx} and K_{xx} consist of the multiples of m_{xx} in G_x , so they must be the trivial subgroup $\{0\}$. The definition of φ_{xx} and the natures of H_{xx} and K_{xx} imply that

$$\varphi_{xx}(\{\ell\}) = \varphi_{xx}(H_{xx} + \ell) = K_{xx} + \ell = \{\ell\},$$

so φ_{xx} is the identity automorphism of $G_x/\{0\}$. Thus, frame condition (i) holds.

Turn now to frame condition (ii). The subgroup H_{yx} is defined to be the set of multiples of m_{yx} in G_y , and the subgroup K_{xy} is defined to be the set of multiples of m_{xy} in G_y . Index condition (ii) says that $m_{yx} = m_{xy}$, so $H_{yx} = K_{xy}$, and similarly, $K_{yx} = H_{xy}$. Furthermore,

$$\varphi_{yx}(H_{xy} + \ell) = K_{yx} + \ell,$$

by the definition of φ_{yx} , while

$$\varphi_{xy}(K_{yx} + \ell) = \varphi_{xy}(H_{xy} + \ell) = K_{xy} + \ell = H_{yx} + \ell,$$

by the preceding remarks and the definition of φ_{xy} . Combine these observations to conclude that $\varphi_{yx} = \varphi_{xy}^{-1}$, which is what frame condition (ii) asserts.

To verify frame condition (iii), just take $s = 0$ in Lemmas 3.4 and 3.5. In a similar fashion, frame condition (iv) follows from Lemmas 3.4–3.6, because

$$\begin{aligned} (\hat{\varphi}_{xy} | \hat{\varphi}_{yz})(H_{xy} \circ H_{xz} + s) &= \hat{\varphi}_{yz}(\hat{\varphi}_{xy}(H_{xy} \circ H_{xz} + s)) \\ &= \hat{\varphi}_{yz}(K_{xy} \circ H_{yz} + s) = K_{xz} \circ K_{yz} + s, \end{aligned}$$

by the definition of the relational composition of two functions, and Lemmas 3.4 and 3.5, while

$$\hat{\varphi}_{xz}(H_{xy} \circ H_{xz} + s) = K_{xz} \circ K_{yz} + s,$$

by Lemma 3.6. \square

It is helpful to visualize index conditions (ii)–(iv) by making a diagram such as the one in Figure 1. Condition (ii) says that each square in the diagram that is on the line $y = x$ (the identity relation) carries the same number as the cardinality of the corresponding group. In the example given in Figure 1, each such square is labeled with the same number 6, because each group is assumed to have cardinality 6, but of course in other examples different groups may have different cardinalities. Condition (iii) says that the diagram must be symmetric across the line $y = x$. To check the validity of condition (iv), it must be shown that any two of any three given indices m_{rs} , m_{rt} , m_{st} have the same greatest common divisor as any other two of the given indices. This can be checked one column at a time. Take two numbers that are in the r th column, m_{rs} and m_{rt} , and then use either row s (if s is to the left of t in the column listing) or row t (if t is to the left of s in the column listing) to locate m_{st} or m_{ts} respectively (it doesn't matter which one because the two indices must be equal), by going along the row to the right until the appropriate column is reached. For a concrete example, observe that in the u th column, $m_{uv} = 3$ and $m_{uy} = 2$. Since y is to the left of v in the column listing, go to the y th row, and move right to the v th column. The entry there is $m_{vy} = 1$. Any two of the three numbers 3, 2, and 1 have the same greatest common divisor, namely 1.

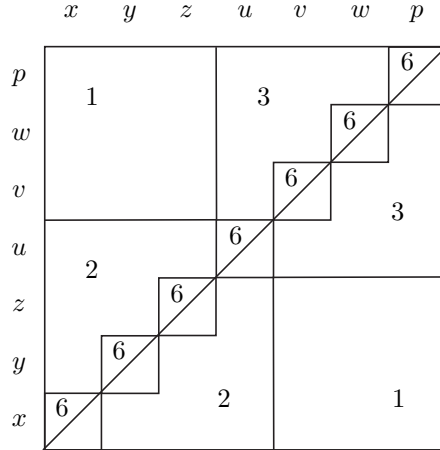


FIGURE 1. A graphical example of verifying the index conditions.

There is a kind of converse to the GCD Theorem that is true.

Theorem 3.9. *Suppose that a group pair $\mathcal{F} = (G, \varphi)$ consists of finite, cyclic groups, with*

$$G = \langle G_x : x \in I \rangle \quad \text{and} \quad \varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle,$$

and satisfies the four frame conditions. If

$$m_{xy} = |G_x / H_{xy}|$$

for every pair (x, y) in \mathcal{E} , where H_{xy} is the kernel of φ_{xy} , then the resulting system

$$m = \langle m_{xy} : (x, y) \in \mathcal{E} \rangle$$

satisfies the index conditions.

Proof. Index condition (i) is satisfied, by the very definition of m_{xy} . As regards index condition (ii), it follows from frame condition (i). In more detail, φ_{xx} is the identity automorphism of $G_x / \{0\}$, by frame condition (i), so $H_{xx} = \{0\}$ and therefore

$$m_{xx} = |G_x / H_{xx}| = |G_x|.$$

Frame condition (ii) says that $\varphi_{yx} = \varphi_{xy}^{-1}$. In particular,

$$H_{yx} = K_{xy} \quad \text{and} \quad K_{yx} = H_{xy},$$

and therefore

$$m_{yx} = |G_y / H_{yx}| = |G_y / K_{xy}| = |G_x / H_{xy}| = m_{xy}.$$

Turn now to index condition (iv). Because m_{xy} is the index of H_{xy} in G_x , the subgroup H_{xy} must consist of the multiples of m_{xy} modulo n_x . Similarly, H_{xz} must consist of the multiples of m_{xz} modulo n_x . It follows from cyclic group theory that the composite group $H_{xy} \circ H_{xz}$ is generated by $d = \gcd(m_{xy}, m_{xz})$, and therefore

$$(1) \quad d = |G_x / (H_{xy} \circ H_{xz})|.$$

Similar arguments show that $K_{xy} \circ H_{yz}$ is generated by $d' = \gcd(m_{xy}, m_{yz})$, so that

$$(2) \quad d' = |G_y / (K_{xy} \circ H_{yz})|,$$

and $K_{xz} \circ K_{yz}$ is generated by $d'' = \gcd(m_{xz}, m_{yz})$, so that

$$(3) \quad d'' = |G_z / (K_{xz} \circ K_{yz})|.$$

The induced isomorphism $\hat{\varphi}_{xy}$ maps the quotient $G_x / (H_{xy} \circ H_{xz})$ isomorphically to the quotient $G_y / (K_{xy} \circ H_{yz})$, by frame condition (iii), so (1) and (2) imply that $d = d'$. Similarly, the induced isomorphism $\hat{\varphi}_{yz}$ maps the quotient $G_y / (K_{xy} \circ H_{yz})$ isomorphically to the quotient $G_z / (K_{xz} \circ K_{yz})$, so (2) and (3) imply that $d' = d''$. Combine these observations with the definitions of d , d' , and d'' to conclude that index condition (iv) holds. \square

4. REGULAR ELEMENTS AND INDICES

We now fix a measurable relation algebra \mathfrak{A} . Thus, the identity element in \mathfrak{A} is the sum of a set I of subidentity atoms, and each atom x in I is measurable in the sense that the square $x; 1; x$ is a sum of non-zero functions below it. These functions are atoms and form a group G_x under the operations of relative multiplication and converse in \mathfrak{A} , with x as the group identity element, by Lemmas 3.2 and 3.3 in [7].

We assume that each such group is finite and cyclic. All elements are assumed to be in \mathfrak{A} . The left and right stabilizers of an element a in \mathfrak{A} are the sets

$$H_a = \{f \in G_x : f; a = a\} \quad \text{and} \quad K_a = \{g \in G_y : a; g = a\},$$

and these stabilizers are (normal) subgroups of G_x and G_y respectively. For measurable atoms x and y , an element $a \leq x; 1; y$ is called *regular* if

$$a; a^\sim = \sum H_a \quad \text{and} \quad a^\sim; a = \sum R_a.$$

It turns out that regular elements have some of the properties of atoms. In particular, every atom is regular, by Partition Lemma 4.11 in [7].

Definition 4.1. For each regular element a , define the *index* of a to be the cardinality of the quotient algebra G_x/H_a , in symbols,

$$\text{index}(a) = |G_x/H_a| = |G_y/K_a|.$$

In other words, the index of a is the number of cosets that the normal subgroup H_a has in G_x , or, equivalently, that the normal subgroup K_a has in G_y . Moreover, for every coset H of H_a there is a uniquely determined coset K of K_a such that

$$H; a = a; K,$$

and conversely, so that the function φ_a from G_x/H_a to G_y/K_a defined by

$$\varphi_a(H) = K \quad \text{if and only if} \quad H; a = a; K$$

is a bijection, and actually a quotient isomorphism.

Lemma 4.2. *If G_y is a cyclic group, and $a \leq x; 1; y$ and $b \leq y; 1; z$ are regular elements, then $a; b \leq x; 1; z$ is a regular element, and*

$$\text{index}(a; b) = \gcd(\text{index}(a), \text{index}(b)).$$

Proof. For notational convenience, assume that G_y is the cyclic group \mathbb{Z}_n under the operation of addition modulo n , and write

$$(1) \quad k = \text{index}(a) = |G_y/K_a| \quad \text{and} \quad \ell = \text{index}(b) = |G_y/H_b|.$$

Thus, K_a and H_b consist of the multiples of k and ℓ modulo n respectively. The complex product

$$K_a; H_b = \{f; g : f \in K_a \text{ and } g \in H_b\}$$

consists of the multiples of $\gcd(k, \ell)$ modulo n , by cyclic group theory. Consequently,

$$(2) \quad |G_y/(K_a; H_b)| = \gcd(k, \ell) = \gcd(\text{index}(a), \text{index}(b)),$$

by cyclic group theory and (1). According to Relative Product Theorem 5.14 in [7], $a; b$ is a regular element, and the isomorphism φ_a from G_x/H_a to G_y/K_a induces an isomorphism $\hat{\varphi}_a$ from $G_x/H_{a; b}$ to $G_y/(K_a; H_b)$. In particular,

$$(3) \quad |G_y/H_{a; b}| = |G_x/(K_a; H_b)|.$$

By definition,

$$(4) \quad \text{index}(a; b) = |G_x/H_{a; b}|.$$

Combine (2)–(4) to arrive at the desired equation. \square

Lemma 4.3. *Let x and y be measurable atoms, and $a, b \leq x; 1; y$ regular elements with $a \leq b$. If $\text{index}(a) = \text{index}(b)$, then $a = b$.*

Proof. The assumption $a \leq b$ implies that

$$(1) \quad H_a \subseteq H_b,$$

by Lemma 4.14 in [7]. The assumption on the indices implies that H_a and H_b have the same number of cosets in G_x . These cosets partition G_x , so it follows from (1) that the inclusion symbol in (1) may be replaced by equality. Apply (the reverse implication of) Lemma 4.14 from [7] to conclude that $a = b$. \square

The next lemma is a known result from cyclic group theory.

Lemma 4.4. *If H and K are subgroups of \mathbb{Z}_n with relatively prime indices h and k , then $\mathbb{Z}_n = H \circ K$, and the cosets of $H \cap K$ in \mathbb{Z}_n are the sets $(H + i) \cap (K + j)$ for $0 \leq i < h$ and $0 \leq j < k$. Each coset has $n/(h \cdot k)$ elements.*

Lemma 4.5. *Let x and y be measurable atoms, and assume that $a, b \leq x; 1; y$ are regular elements with $a \cdot b \neq 0$. If $\text{index}(a)$ and $\text{index}(b)$ are relatively prime, then*

$$\text{index}(a \cdot b) = \text{index}(a) \cdot \text{index}(b).$$

Proof. Product Theorem 4.13 from [7] and the hypothesis that $a \cdot b \neq 0$ imply that

$$(1) \quad H_{a \cdot b} = H_a \cap H_b.$$

The cosets of H_a and H_b have the form $H_a + i$ and $H_b + j$ for $0 \leq i < \text{index}(a)$ and $0 \leq j < \text{index}(b)$ respectively. Apply Lemma 4.4 (with H_a and H_b in place of H and K respectively), and use the assumption that the indices of a and b are relatively prime, to see that

$$\langle (H_a + i) \cap (H_b + j) : i < \text{index}(a) \text{ and } j < \text{index}(b) \rangle$$

is a coset system for (1) in G_x , consisting of $\text{index}(a) \cdot \text{index}(b)$ cosets, and each coset has ℓ elements, where

$$|G_x| = n_x = \text{index}(a) \cdot \text{index}(b) \cdot \ell.$$

Consequently, the index of (1) in G_x is

$$(2) \quad |G_x / (H_a \cap H_b)| = \text{index}(a) \cdot \text{index}(b),$$

by the definition of the index. Apply (1), (2), and the definition of the index of an element to conclude that

$$\text{index}(a \cdot b) = |G_x / H_{a \cdot b}| = |G_x / (H_a \cap H_b)| = \text{index}(a) \cdot \text{index}(b).$$

\square

Corollary 4.6. *Let x and y be measurable atoms. If $a_i \leq x; 1; y$ for $i = 1, \dots, n$ are regular elements with pairwise relatively prime indices, and if $\prod_{i=1}^n a_i \neq 0$, then*

$$\text{index}\left(\prod_{i=1}^n a_i\right) = \prod_{i=1}^n \text{index}(a_i).$$

The proof is by induction on n . The details are left to the reader. Observe that the symbol \prod is being used in two different ways in the preceding corollary. In its first occurrence, it denotes the Boolean operation of multiplication on finite sequences of elements in a Boolean algebra. In its second occurrence, it denotes the arithmetic operation of multiplication on finite sequences of natural numbers. This double usage of the symbol is very common and should not cause readers any confusion. A similar remark applies to the use of the symbol \cdot in Lemma 4.5.

Lemma 4.7. *Let x and y be measurable atoms, and $a, b \leq x; 1; y$ regular elements. If the indices of a and b are relatively prime, then $a \cdot b$ is a regular element below $x; 1; y$.*

Proof. The assumption on the indices of a and b implies that the left stabilizers H_a and H_b have relatively prime indices, and therefore

$$(1) \quad H_a; H_b = H_b; H_a = G_x,$$

by Lemma 4.4. The definition of H_b as the left stabilizer of b means that

$$(2) \quad H_b; b = b.$$

Consequently,

$$(3) \quad \sum H_a; b = \sum H_a; H_b; b = \sum G_x; b = (x; 1; x); b = x; 1; y,$$

by (2), (1), the fact that $\sum G_x = x; 1; x$, by Corollary 3.5 in [7], and Lemma 2.3(ii). Similarly,

$$(4) \quad \sum H_b; a = \sum H_b; H_a; a = \sum G_x; a = (x; 1; x); a = x; 1; y.$$

Combine (3) and (4) to arrive at

$$(5) \quad \sum H_a; b = \sum H_b; a.$$

Use the assumption that a and b are regular elements, together with the definition of such elements, and apply it to (5) to obtain

$$(6) \quad a; a^\smile; b = b; b^\smile; a.$$

Part (i) of Product Theorem 4.18 in [7] says that the condition in (6) is equivalent to the assertion that $a \cdot b \neq 0$. Apply Product Theorem 4.13 from [7] to conclude that $a \cdot b$ is a regular element below $x; 1; y$. \square

Corollary 4.8. *Let x and y be measurable atoms. If $a_i \leq x; 1; y$ for $i = 1, \dots, n$ are regular elements with pairwise relatively prime indices, then $\prod_{i=1}^n a_i$ is a regular element below $x; 1; y$.*

The proof is by induction on n . The details are left to the reader.

5. THE REPRESENTATION THEOREM

We continue with the assumption that \mathfrak{A} is a measurable relation algebra. The assumption that the groups are finite and cyclic implies that the algebra \mathfrak{A} is finitely measurable, and hence automatically atomic, by Theorem 8.3 in [7]. The goal is to show that \mathfrak{A} is essentially isomorphic to a group relation algebra $\mathfrak{G}[\mathcal{F}]$ for one of the group frames \mathcal{F} constructed in GCD Theorem 3.8. This means that we must show that the completion of \mathfrak{A} (in the sense of the minimal complete extension of \mathfrak{A}) is isomorphic to $\mathfrak{G}[\mathcal{F}]$. Scaffold Representation Theorem 7.4 in [7] says that an atomic measurable relation algebra \mathfrak{A} is essentially isomorphic to some group relation algebra if and only if it has a scaffold. A *scaffold* in \mathfrak{A} is a system $\langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ of atoms in \mathfrak{A} that satisfies the following conditions for all (x, y) and (y, z) in \mathcal{E} .

- (i) $a_{xx} = x$.
- (ii) $a_{yx} = a_{xy}^\smile$.

$$(iii) \ a_{xz} \leq a_{xy}; a_{yz}.$$

Thus, to prove the desired representation theorem for \mathfrak{A} , it suffices to construct a scaffold in \mathfrak{A} .

Fix measurable atoms x and y in \mathfrak{A} with (x, y) in \mathcal{E} , and consider a regular element $a \leq x; 1; y$. Regular elements are always non-zero, by Lemma 4.4 in [7], so $a \neq 0$. If functions f and g belong to the same coset H of H_a , then the left translations $f; a$ and $g; a$ of a are equal, and if they belong to different cosets of H_a , then $(f; a) \cdot (g; a) = 0$, by Lemma 4.6(iii) in [7]. Thus, it makes sense to write $H; a$ whenever H is a coset of H_a , and this just denotes the element $f; a$ for some (any) element f in H . The left translations $b = H; a$ by cosets H of H_a are mutually disjoint, regular elements below $x; 1; y$ with the same normal stabilizer H_a as a , and in fact these left translations form a partition of $x; 1; y$, by Partition Lemma 4.9 in [7]. Similar remarks apply to the right translations $a; K$ of a by cosets K of K_a (in G_y).

Since \mathfrak{A} is atomic, there must be an atom below $x; 1; y$, and such atoms are regular elements with the same stabilizer, by Partition Lemma 4.11 in [7]. Write H_{xy} and K_{xy} for the left and right stabilizer of these atoms, and for a fixed atom a , write φ_{xy} for the quotient isomorphism φ_a . The choice of φ_{xy} is dependent on a , but a system of atoms can be chosen with the following properties (see, for example, the remarks in Section 7 of [7]).

- (P1) $H_{xx} = \{x\}$, and φ_{xx} is the identity isomorphism of G_x/H_{xx} .
- (P2) $H_{yx} = K_{xy}$, $K_{yx} = H_{xy}$, and $\varphi_{yz} = \varphi_{xy}^{-1}$.
- (P3) $\varphi_{xy}[H_{xy}; H_{xz}] = K_{xy}; H_{yz}$, $\varphi_{yz}[K_{xy}; H_{yz}] = K_{xz}; K_{yz}$, and $\varphi_{xz}[H_{xy}; H_{xz}] = K_{xz}; K_{yz}$.
- (P4) $|G_x/(H_{xy}; H_{xz})| = |G_y/(K_{xy}; H_{yz})| = |G_z/(K_{xz}; K_{yz})|$.

In fact, the isomorphism φ_{xy} induces an isomorphism $\hat{\varphi}_{xy}$ from $G_x/(H_{xy}; H_{xz})$ to $G_y/(K_{xy}; H_{yz})$, while φ_{yz} induces an isomorphism $\hat{\varphi}_{yz}$ from $G_y/(K_{xy}; H_{yz})$ to $G_z/(K_{xz}; K_{yz})$. Property (P4) is an immediate consequence of this observation.

Put $m_{xy} = |G_x/H_{xy}|$.

Lemma 5.1 (Index Lemma). *Suppose (x, y) and (y, z) are pairs of measurable atoms in \mathcal{E} .*

- (i) $m_{xx} = |G_x/H_{xx}| = |G_x/\{x\}| = |G_x|$.
- (ii) $m_{yx} = m_{xy}$.
- (iii) $\gcd(m_{xy}, m_{xz}) = \gcd(m_{xy}, m_{yz}) = \gcd(m_{xz}, m_{yz})$.

Proof. Property (P1) and the definition of m_{xx} imply that

$$m_{xx} = |G_x/H_{xx}| = |G_x/\{0\}| = |G_x|.$$

Property (P2) and the definitions of m_{xy} and m_{yx} imply that

$$m_{yx} = |G_y/H_{yx}| = |G_y/K_{xy}| = |G_x/H_{xy}| = m_{xy}.$$

Since H_{xy} and H_{xz} have indices m_{xy} and m_{xz} , they are respectively generated by (copies of) the integers m_{xy} and m_{xz} modulo n_x . Consequently, the group composition $H_{xy}; H_{xz}$ (under the operation of relative multiplication in \mathfrak{A}) is generated by the element $\gcd(m_{xy}, m_{xz})$, so that

$$(1) \quad \gcd(m_{xy}, m_{xz}) = |G_x/(H_{xy}; H_{xz})|.$$

Similarly,

$$(2) \quad \gcd(m_{xy}, m_{yz}) = |G_y / (K_{xy}; H_{yz})|.$$

$$(3) \quad \gcd(m_{xz}, m_{yz}) = |G_z / (K_{xz}; K_{yz})|.$$

Combine (1)–(3) with property (P4) to arrive at (iii). \square

Fix a prime number p for the next definition and two lemmas, and in terms of this prime, define a binary relation \sim_k as follows.

Definition 5.2. For x and y in I , define $x \sim_k y$ if and only if $x = y$, or (x, y) is in \mathcal{E} and p^k divides m_{xy} .

Lemma 5.3. *The relation \sim_k is an equivalence relation on the set I .*

Proof. The relation is automatically reflexive, by its very definition. For symmetry, suppose that $x \sim_k y$ and $x \neq y$. In this case (x, y) belongs to \mathcal{E} and p^k divides m_{xy} . The relation \mathcal{E} is symmetric, so it contains (y, x) , and $m_{yx} = m_{xy}$, so p^k divides m_{yx} . Thus, $y \sim_k x$, by Definition 5.2.

Turn now to transitivity. Assume that $x \sim_k y$ and $y \sim_k z$. If two of these atoms are equal, then the proof of transitivity is trivial, so suppose that all three atoms are distinct. The hypotheses and Definition 5.2 imply that p^k divides m_{xy} and m_{yz} , so it divides their greatest common divisor. Since

$$\gcd(m_{xz}, m_{yz}) = \gcd(m_{xy}, m_{yz})$$

by Index Lemma 5.1, it follows that p^k divides m_{xz} . Also, \mathcal{E} is transitive, so the pair (x, z) is in \mathcal{E} . Therefore, $x \sim_k z$, by Definition 5.2. \square

Lemma 5.4. *For each prime p , there is a system of elements*

$$\langle a_{xy}^k : k \geq 0 \text{ and } x \sim_k y \rangle$$

with the following properties whenever $x \sim_k y$ and $y \sim_k z$.

- (i) a_{xy}^k is a regular element below $x; 1; y$, and $a_{xy}^k = x; 1; y$ when $x \neq y$ and $k = 0$.
- (ii) If $x = y$, then $a_{xy}^k = x$.
- (iii) If $x \neq y$, then $\text{index}(a_{xy}^k) = p^k$.
- (iv) $a_{yx}^k = (a_{xy}^k)^\smile$.
- (v) $a_{xz}^k \leq a_{xy}^k; a_{yz}^k$, and equality holds when $x \neq z$.
- (vi) $a_{xy}^k \leq a_{xy}^{k-1}$ for $k \geq 1$.

Proof. The construction is by induction on k starting at $k = 0$. In this case, the definition of a_{xy}^k is dictated by the first two conditions:

$$(1) \quad a_{xy}^0 = \begin{cases} x & \text{if } x = y, \\ x; 1; y & \text{if } x \neq y. \end{cases}$$

It is not difficult to verify that in this case properties (i)–(vi) hold. If $x = y$, then the element $x \leq x; 1; x$ is regular with left and right stabilizers $\{x\}$. If $x \neq y$, then $x; 1; y$ is regular with left and right stabilizers G_x and G_y respectively. Consequently,

$$\text{index}(a_{xy}^0) = |G_x / G_x| = 1 = p^0,$$

so properties (i)–(iii) hold.

For property (iv), observe that if $x = y$, then

$$a_{yx}^0 = y = x = x^\sim = (a_{xy}^k)^\sim,$$

by Lemma 2.1(v). On the other hand, if $x \neq y$, then

$$a_{yx}^0 = y; 1; x = (x; 1; y)^\sim = (a_{xy}^0)^\sim,$$

by Lemma 2.3(i).

The verification of property (v) breaks down into cases. If $x = y$, then

$$a_{xy}^k; a_{yz}^k = x; a_{yz}^k = x; a_{xz}^k = a_{xz}^k,$$

by condition (ii), the assumption that $x = y$, and Lemma 2.3(iii). A similar argument applies if $y = z$. If $x = z$, then

$$a_{xy}^k; a_{yz}^k = a_{xy}^k; a_{yx}^k = a_{xy}^k; (a_{xy}^k)^\sim = \sum H_{xy} \geq x = a_{xx}^k = a_{xz}^k,$$

by the assumption that $x = z$, condition (iv) (which has already been shown to hold when $k = 0$), the regularity of a_{xy}^k from condition (i) (which has already been shown to hold when $k = 0$), the fact that x is in its left stabilizer H_{xy} , condition (ii) (which has already been shown to hold when $k = 0$), and the assumption $x = z$. If x , y , and z are pairwise distinct, then

$$a_{xy}^k; a_{yz}^k = (x; 1; y); (y; 1; z) = x; 1; z = a_{xz}^k,$$

by the definition in (1) and Lemma 2.3(ii).

Condition (vi) holds vacuously.

Assume now that a_{xy}^{k-1} has been defined for all pairs (x, y) with $x \sim_{k-1} y$ so that conditions (i)–(vi) hold (with $k-1$ in place of k , and $k \geq 1$). For each measurable atom x in I , choose a representative \bar{x} of the equivalence class x / \sim_k . Thus,

$$(2) \quad x \sim_k y \quad \text{if and only if} \quad \bar{x} = \bar{y}.$$

The next step is to construct for each element $y \sim_k x$ an element $c_{y\bar{x}} \leq y; 1; \bar{x}$ as follows. If $y = \bar{x}$, put

$$(3) \quad c_{y\bar{x}} = \bar{x}.$$

If $y \neq \bar{x}$, then write $b = a_{y\bar{x}}^{k-1}$, and fix an atom $d \leq b$. Such an atom exists because b is a regular element, by the induction hypothesis for condition (i), and hence b is non-zero. The atomicity of \mathfrak{A} implies that every non-zero element is above an atom. The element b has index p^{k-1} by the induction hypothesis for condition (iii), so the subgroup H_b is generated by p^{k-1} . Let L be the subgroup of G_x generated by p^k . The element p^{k-1} generates H_b , and p^k is a multiple of p^{k-1} , so p^k belongs to H_b , and therefore L is included in H_b . On the other hand, $m_{y\bar{x}}$ generates $H_{y\bar{x}}$, and p^k divides $m_{y\bar{x}}$, so $H_{y\bar{x}}$ must be included in L . Thus,

$$(4) \quad H_{y\bar{x}} \subseteq L \subseteq H_b.$$

Observe that

$$(5) \quad |G_x / L| = p^k,$$

since p^k generates L .

Put

$$(6) \quad c_{y\bar{x}} = \sum L; d.$$

Because L is a subgroup of G_x , and d is an atom satisfying $d \leq b \leq y; 1; \bar{x}$ (the second inequality uses the induction hypothesis for condition (i)), the element $c_{y\bar{x}}$

is a regular element below $y; 1; \bar{x}$, and its left stabilizer is L , by Theorem 9.1 in [7]. Consequently,

$$(7) \quad \text{index}(c_{y\bar{x}}) = |G_x/L| = p^k,$$

by (5). The preceding observations and (3) show that conditions (i)–(iii) of the lemma hold with $c_{y\bar{x}}$ in place of $a_{y\bar{x}}^k$.

For $y \neq \bar{x}$, define

$$(8) \quad c_{\bar{x}y} = c_{y\bar{x}}^\smile,$$

and observe that since the converse of a regular element below $y; 1; \bar{x}$ is a regular element below $\bar{x}; 1; y$, with the left and right stabilizers reversed, by Converse Theorem 5.13 from [7], the element defined in (8) must be regular, below $\bar{x}; 1; y$, and have the same index as $c_{y\bar{x}}$, so that

$$(9) \quad \text{index}(c_{\bar{x}y}) = p^k,$$

by (7).

For arbitrary x and y in I with $x \sim_k y$, define

$$(10) \quad a_{xy}^k = \begin{cases} x & \text{if } x = y, \\ c_{x\bar{x}}; c_{\bar{x}y} & \text{if } x \neq y. \end{cases}$$

Observe that a_{xy}^k is well defined by (2). The relative product of a regular element below $x; 1; \bar{x}$ and a regular element below $\bar{x}; 1; y$ is a regular element below $x; 1; y$, by Relative Product Theorem 5.16 in [7], so condition (i) of the lemma is satisfied when $x \neq y$, and it is trivially satisfied when $x = y$. Condition (ii) is automatically satisfied, by (10).

Turn to the verification of condition (iii). Assume $x \sim_k y$ are distinct. Use (10) and Lemma 4.2 (with $c_{x\bar{x}}$ and $c_{\bar{x}y}$ in place of a and b respectively) to obtain

$$(11) \quad \text{index}(a_{xy}^k) = \text{index}(c_{x\bar{x}}; c_{\bar{x}y}) = \gcd(\text{index}(c_{x\bar{x}}), \text{index}(c_{\bar{x}y})).$$

The value of $\text{index}(c_{x\bar{x}})$ is either p^k or $|G_x|$ according to whether $x \neq \bar{x}$ or $x = \bar{x}$, by (7) and (3), and similarly for $c_{\bar{x}y}$, by (7), (3), and (8). At least one of them must be p^k since $y \neq x$, so the value of (11) is p^k . This completes the verification of condition (iii).

Take up now condition (iv). If $x = y$, then

$$a_{yx}^k = y = x = x^\smile = (a_{xy}^k)^\smile,$$

by (10) and Lemma 2.1(v). If $x \sim_k y$ are distinct, then

$$a_{yx}^k = (c_{y\bar{y}}; c_{\bar{y}x}) = c_{\bar{y}y}^\smile; c_{x\bar{y}}^\smile = (c_{x\bar{y}}; c_{\bar{y}y})^\smile = (c_{x\bar{x}}; c_{\bar{x}y})^\smile = (a_{xy}^k)^\smile,$$

by (10), (8), the second involution law, (2), and (10).

The next verification is of condition (v). Assume that $x \sim_k y \sim_k z$, and consider first the cases when at least two of the three atoms are equal. If $x = y$, then

$$a_{xy}^k; a_{yz}^k = x; a_{yz}^k = x; a_{xz}^k = a_{xz}^k,$$

by (10), the assumption that $x = y$, and Lemma 2.3(iii). The argument when $y = z$ is similar. If $x = z$, then

$$a_{xy}^k; a_{yz}^k = a_{xy}^k; a_{yx}^k = a_{xy}^k; (a_{xy}^k)^\smile = \sum H_{xy} \geq x = a_{xz}^k,$$

by the assumption that $x = z$, condition (iv), the regularity of a_{xy}^k , which is ensured by condition (i), the fact that H_{xy} contains x , monotony, and (10).

Assume now that the atoms x , y , and z are mutually distinct. The element $c_{\bar{y}y}$ is regular, and it is the converse of $c_{y\bar{y}}$, by (8), so

$$(12) \quad c_{\bar{y}y}; c_{y\bar{y}} = c_{y\bar{y}}^\smile; c_{y\bar{y}} = \sum K_{y\bar{y}} \geq \bar{y}$$

(the last step uses monotony and the fact that \bar{y} is in $K_{y\bar{y}}$). Consequently,

$$\begin{aligned} a_{xy}^k; a_{yz}^k &= (c_{x\bar{x}}; c_{\bar{x}y}); (c_{y\bar{y}}; c_{\bar{y}z}) = (c_{x\bar{x}}; c_{\bar{y}y}); (c_{y\bar{y}}; c_{\bar{y}z}) \\ &= c_{x\bar{x}}; (c_{\bar{y}y}; c_{y\bar{y}}); c_{\bar{y}z} \geq c_{x\bar{x}}; \bar{y}; c_{\bar{y}z} = c_{x\bar{x}}; c_{\bar{y}z} = c_{x\bar{x}}; c_{\bar{x}z} = a_{xz}^k, \end{aligned}$$

by (10), (2), the associative law, (12) and monotony, Lemma 2.3(iii), (2), and (10). This argument shows that

$$(13) \quad a_{xz}^k \leq a_{xy}^k; a_{yz}^k.$$

On the other hand,

$$\begin{aligned} (14) \quad \text{index}(a_{xy}^k; a_{yz}^k) &= \text{gcd}(\text{index}(a_{xy}^k), \text{index}(a_{yz}^k)) \\ &= \text{gcd}(p^k, p^k) = p^k = \text{index}(a_{xz}^k), \end{aligned}$$

by condition (i), Lemma 4.2, condition (iii), and the assumption on x , y , and z . Use condition (i), (13), (14), and Lemma 4.3 (with a_{xz}^k and $a_{xy}^k; a_{yz}^k$ in place of a and b) to conclude that

$$a_{xz}^k = a_{xy}^k; a_{yz}^k.$$

Turn finally to the verification of condition (vi). If $x = y$, then

$$a_{xy}^k = x = a_{xy}^{k-1},$$

by (10) and the induction hypothesis for condition (ii), so condition (vi) holds in this case. Assume now that $x \sim_k y$ are distinct. At least one of x and y must be different from \bar{x} , say it is y . If $x = \bar{x}$, then

$$(15) \quad a_{xy}^k = c_{x\bar{x}}; c_{\bar{x}y} = \bar{x}; c_{\bar{x}y} = c_{\bar{x}y} = c_{y\bar{x}}^\smile,$$

by (10), (3), Lemma 2.3(iii), and (8). Write $b = a_{y\bar{x}}^{k-1}$. The element $c_{y\bar{x}}$ is defined to be $\sum L; d$, where d is some atom below b , and L satisfies the inclusions in (4). The second inclusion in (4) implies that $L; d \subseteq H_b; d$, and therefore

$$(16) \quad c_{y\bar{x}} = \sum L; d \leq \sum H_b; d \leq \sum H_b; b = b = a_{y\bar{x}}^{k-1},$$

by (6), monotony, the fact that $d \leq b$, the definition of H_b as the stabilizer of b , and the definition of b . Apply (16), monotony, and the induction hypothesis for condition (iv) to arrive at

$$(17) \quad c_{y\bar{x}}^\smile \leq (a_{y\bar{x}}^{k-1})^\smile = a_{\bar{x}y}^{k-1}.$$

With the help of (15), (17), Lemma 2.3(iii), the assumption that $x = \bar{x}$, the induction hypotheses for (ii) and (v), and the assumption that $y \neq \bar{x}$, conclude that

$$a_{xy}^k = c_{y\bar{x}}^\smile \leq a_{\bar{x}y}^{k-1} = \bar{x}; a_{\bar{x}y}^{k-1} = x; a_{\bar{x}y}^{k-1} = a_{x\bar{x}}^{k-1}; a_{\bar{x}y}^{k-1} = a_{xy}^{k-1}.$$

Consider finally the case when both x and y are different from \bar{x} . An argument analogous to that of (16) shows that

$$(18) \quad c_{x\bar{x}} \leq a_{x\bar{x}}^{k-1}.$$

Also, it follows from (17) and (8) that

$$(19) \quad c_{\bar{x}y} \leq a_{\bar{x}y}^{k-1}.$$

Compute:

$$a_{xy}^k = c_{x\bar{x}}; c_{\bar{x}y} \leq a_{x\bar{x}}^{k-1}; a_{\bar{x}y}^{k-1} = a_{xy}^{k-1},$$

by (10), (18), (19), monotony, and the induction hypothesis for (v) (with \bar{x} and y in place of y and z respectively). This completes the proof of the lemma. \square

For each pair (x, y) in \mathcal{E} and each natural number $k \geq 0$ such that $x \sim_k y$, an element a_{xy}^k has been constructed in Lemma 5.4 such that the system of these elements possesses certain properties. The next step is to use these elements and properties in order to construct a scaffold. Fix a pair (x, y) in \mathcal{E} , and let

$$m_{xy} = p_1^{k_1} \cdot \dots \cdot p_n^{k_n}$$

be the decomposition of m_{xy} into distinct primes. If $m_{xy} = 1$, write $m_{xy} = 2^0$ for the prime decomposition. Thus, for each index $i = 1, \dots, n$, we have $x \sim_{k_i} y$, and $p_i^{k_i}$ is the largest power of p_i that divides m_{xy} when $x \neq y$.

We now change notation a bit by letting the prime numbers p_i vary as i varies over $1, \dots, n$, and writing $a_{xy}^{k_i}$ to denote the element constructed in Lemma 5.4 using the prime $p = p_i$ and the natural number $k = k_i$. Thus, in contrast to the lemma, we let k_i and k_j represent powers of different primes, namely p_i and p_j , in the notations $a_{xy}^{k_i}$ and $a_{xy}^{k_j}$.

Definition 5.5. For each pair (x, y) in \mathcal{E} , write

$$a_{xy} = a_{xy}^{k_1} \cdot \dots \cdot a_{xy}^{k_n} = \prod_{i=1}^n a_{xy}^{k_i},$$

where $m_{xy} = p_1^{k_1} \cdot \dots \cdot p_n^{k_n}$ is the prime decomposition of m_{xy} .

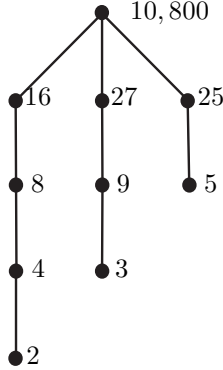


FIGURE 2. Diagram of the levels of construction of the scaffold.

Theorem 5.6 (Scaffold Theorem). *The system $\langle a_{xy} : (x, y) \in \mathcal{E} \rangle$ is a scaffold in \mathfrak{A} .*

Proof. The first task is to prove that

$$(1) \quad \text{index}(a_{xy}) = m_{xy}$$

for all $(x, y) \in \mathcal{E}$. Consider the case when $x \neq y$. The element $a_{xy}^{k_i}$ is regular and below $x; 1; y$, and

$$(2) \quad \text{index}(a_{xy}^{k_i}) = p^{k_i},$$

by Lemma 5.4(i),(iii). In particular, these indices are relatively prime to one another for distinct indices i . There are now two subcases to consider. If $m_{xy} > 1$, then

$$(3) \quad a_{xy} = \prod_{i=1}^n a_{xy}^{k_i}$$

is a product of regular elements below $x; 1; y$ with mutually relatively prime indices, so it is a regular element below $x; 1; y$, and

$$(4) \quad \text{index}(a_{xy}) = \prod_{i=1}^n \text{index}(a_{xy}^{k_i}) = \prod_{i=1}^n p^{k_i} = m_{xy},$$

by Corollary 4.6 and (2). On the other hand, if $m_{xy} = 1$, then

$$(5) \quad a_{xy} = x; 1; y,$$

by Lemma 5.4(i), so the left stabilizer of a_{xy} is G_x , and consequently

$$(6) \quad \text{index}(a_{xy}) = |G_x/G_x| = 1 = m_{xy}.$$

In the remaining case, $x = y$, so that $a_{xy}^{k_i} = x$ for each i , by Lemma 5.4(ii), and therefore

$$(7) \quad a_{xy} = a_{xx} = x,$$

by (3). The left stabilizer of x is $\{x\}$, by the group identity law, so

$$(8) \quad \text{index}(a_{xx}) = \text{index}(x) = |G_x/\{x\}| = |G_x| = m_{xx},$$

by (7), Definition 4.1, and Lemma 5.1(i). Combine (4), (6), and (8) to conclude that (1) holds in all cases.

The next task is to prove that the element a_{xy} is an atom. As was pointed out earlier, the algebra \mathfrak{A} is finitely measurable, and hence atomic. Since a_{xy} is regular, it is non-zero, and therefore there must be an atom $d \leq a_{xy}$. The atom d is regular, by Corollary 4.7 in [7], and its left stabilizer is H_{xy} , by Corollary 4.16 in [7]. Use the definition of the index of an element, the definition of m_{xy} , and (1) to arrive at

$$\text{index}(d) = |G_x/H_{xy}| = m_{xy} = \text{index}(a_{xy}).$$

Apply Lemma 4.3 to conclude that $d = a_{xy}$, and hence that a_{xy} is an atom with left stabilizer H_{xy} .

It remains to verify the three scaffold conditions. The first one holds by (7). To verify the second scaffold condition, it suffices to check the case when $x \neq y$ (see Theorem 4.4 in [3]). If $m_{xy} > 1$, then Lemma 5.4(iv) ensures that

$$(9) \quad a_{yx}^{k_i} = (a_{xy}^{k_i})^\sim,$$

for each i , and consequently,

$$(10) \quad a_{xy}^\sim = (\prod_i a_{xy}^{k_i})^\sim = \prod_i (a_{xy}^{k_i})^\sim = \prod_i a_{yx}^{k_i} = a_{yx},$$

by (3) and the assumption that $m_{xy} > 1$, Lemma 2.1(ii), (9), and (3) (with y and z in place of x and y respectively). If $m_{xy} = 1$, then $m_{yx} = 1$, and

$$(11) \quad a_{xy}^\sim = (x; 1; y)^\sim = y; 1; x = a_{yx},$$

by (5) and Lemma 2.3(i). Thus, the second scaffold condition holds in all cases, by (10) and (11).

It remains to verify the third scaffold condition. Consider pairs (x, y) and (y, z) in \mathcal{E} . If $x = y$, then

$$(12) \quad a_{xz} = x; a_{xz} = a_{xx}; a_{xz} = a_{xy}; a_{yz},$$

by Lemma 2.3(iii), (7), and the assumption $x = y$. A similar argument applies if $y = z$. If $x = z$, then

$$(13) \quad a_{xz} = a_{xx} = x \leq \sum H_{xy} = a_{xy}; a_{xy}^\sim = a_{xy}; a_{yx} = a_{xy}; a_{yz},$$

by the assumption $x = z$, (7), the fact that x belongs to the left stabilizer H_{xy} , monotony, the regularity of the atom a_{xy} , and scaffold condition (ii), which has already been shown to hold.

Assume now that x, y , and z are all distinct. If $m_{xy} = 1$, then (5) holds, and therefore

$$(14) \quad a_{xy}; a_{yz} = x; 1; y; a_{yz} = x; 1; z \geq a_{xz},$$

by (5), Lemma 2.3(ii), and Lemma 5.4(i). A similar argument applies if $m_{yz} = 1$. We may therefore assume that $m_{xy} > 1$ and $m_{yz} > 1$. Thus, m_{xy} and m_{yz} each have at least one prime in their prime decompositions. Suppose

$$(15) \quad m_{xy} = p_1^{k_1} \cdot \dots \cdot p_n^{k_n} = \prod_{i=1}^n p_i^{k_i},$$

so that a_{xy} has the form (3). Forming the product of a_{xy} with the element $x; 1; y$ does not change the value of a_{xy} , since a_{xy} is below this element, by Lemma 5.4(i).

This amounts to forming the product of a_{xy} with elements of the form $a_{xy}^{\ell_j}$ in which $\ell_j = 0$, by Lemma 5.4(i). The same reasoning applies to the values of a_{yz} and a_{xz} , so by multiplying such zero powers of primes into the factorizations of m_{xy} , m_{yz} , and m_{xz} , we may assume that

$$(16) \quad m_{yz} = p_1^{\ell_1} \cdot \dots \cdot p_n^{\ell_n} = \prod_{i=1}^n p_i^{\ell_i} \quad \text{and} \quad m_{xz} = p_1^{j_1} \cdot \dots \cdot p_n^{j_n} = \prod_{i=1}^n p_i^{j_i},$$

that is to say, the same primes p_i occur in all three factorizations, some of them raised to the zeroth power. Consequently,

$$(17) \quad a_{yz} = \prod_{i=1}^n a_{yz}^{\ell_i} \quad \text{and} \quad a_{xz} = \prod_{i=1}^n a_{xz}^{j_i}.$$

Write

$$(18) \quad s_i = \min\{k_i, \ell_i\}$$

for each $i = 1, \dots, n$, and observe that

$$(19) \quad \gcd(m_{xy}, m_{yz}) = \gcd(\prod_i p_i^{k_i}, \prod_i p_i^{\ell_i}) = \prod_i p_i^{s_i}.$$

Since

$$\gcd(m_{xy}, m_{yz}) = \gcd(m_{xy}, m_{xz}),$$

by Lemma 5.1(iii), it follows from (19) that

$$(20) \quad p_i^{s_i} \text{ divides } m_{xz}.$$

The definition of a_{xy} implies that k_i is the largest natural number such that $x \sim_{k_i} y$. Similarly, ℓ_i is the largest natural number such that $y \sim_{\ell_i} z$. It follows from (18) and the definition of the relation \sim_{s_i} that $x \sim_{s_i} y$ and $y \sim_{s_i} z$, so

$$(21) \quad x \sim_{s_i} z,$$

by transitivity. Since j_i is the largest natural number such that $x \sim_{j_i} z$, it follows from (20) and (21) that $s_i \leq j_i$, and therefore

$$(22) \quad a_{xz}^{j_i} \leq a_{xz}^{s_i},$$

by Lemma 5.4(vi). Use (17) and (22) to conclude that

$$(23) \quad a_{xz} = \prod_i a_{xz}^{j_i} \leq \prod_i a_{xz}^{s_i}.$$

The element a_{xz} is an atom, so in particular, (23) implies that

$$(24) \quad a_{xz}^{s_i} \neq 0.$$

Use Corollary 4.6, Lemma 5.4(iii), and (19) to arrive at

$$(25) \quad \text{index}(\prod_i a_{xz}^{s_i}) = \prod_i \text{index}(a_{xz}^{s_i}) = \prod_i p_i^{s_i} = \gcd(m_{xy}, m_{yz}).$$

The indices of a_{xy} and a_{yz} are m_{xy} and m_{yz} respectively, by (4). Use this observation and Lemma 4.2 to write

$$(26) \quad \gcd(m_{xy}, m_{yz}) = \gcd(\text{index}(a_{xy}), \text{index}(a_{yz})) = \text{index}(a_{xy}; a_{yz}).$$

Combine (26) with (25) to arrive at

$$(27) \quad \text{index}(a_{xy}; a_{yz}) = \text{index}(\prod_i a_{xz}^{s_i}).$$

Use (3), (17), monotony, (18), and Lemma 5.4(vi) to obtain

$$(28) \quad a_{xy}; a_{yz} = (\prod_i a_{yz}^{k_i}); (\prod_i a_{yz}^{\ell_i}) \leq \prod_i (a_{yz}^{k_i}; a_{yz}^{\ell_i}) \leq \prod_i a_{yz}^{s_i}.$$

In view of (28), Lemma 4.3 (with $a_{xy}; a_{yz}$ and $\prod_i a_{yz}^{s_i}$ in place of a and b respectively) may be applied to (27), and then (23) may be invoked, to conclude that

$$a_{xy}; a_{yz} = \prod_i a_{xz}^{s_i} \geq a_{xz}.$$

This completes the verification of the third scaffold condition and hence the proof of the theorem. \square

Here, finally, is the representation theorem for measurable relation algebras with finite cyclic groups.

Theorem 5.7 (Representation Theorem). *If \mathfrak{A} is a measurable relation algebra, and if, for each measurable atom x , the group G_x is finite and cyclic, then \mathfrak{A} is essentially isomorphic to one of the cyclic group relation algebras constructed in GCD Theorem 3.8. Hence, \mathfrak{A} is completely representable.*

Proof. Here is a summary of the strategy that was outlined at the beginning of the section. The groups G_x are all assumed to be finite, so \mathfrak{A} is finitely measurable and therefore atomic. Using this fact, a scaffold is constructed in \mathfrak{A} , by Scaffold Theorem 5.6. A measurable relation algebra with a scaffold is essentially isomorphic to a full group relation algebra, by Scaffold Representation Theorem 7.4 in [7].

To see which full group relation algebra, let I be the set of measurable atoms in \mathfrak{A} , and for each atom x in I , let G_x be the group of non-zero functions below the square $x; 1; x$. Take \mathcal{E} to be the set of pairs (x, y) such that $x; 1; y \neq 0$. Fix a scaffold

$$(1) \quad \langle a_{xy} : (x, y) \in \mathcal{E} \rangle$$

in \mathfrak{A} . For each pair (x, y) in \mathcal{E} , take H_{xy} and K_{xy} to be the left and right stabilizers of the atom a_{xy} . The function φ_{xy} from G_x/H_{xy} to G_y/K_{xy} defined for cosets H of H_{xy} and K of K_{xy} by

$$\varphi_{xy}(H) = K \quad \text{if and only if} \quad H; a_{xy} = a_{xy}; K$$

is an isomorphism. The group pair $\mathcal{F} = (G, \varphi)$ consisting of the systems

$$G = \langle G_x : x \in I \rangle \quad \text{and} \quad \varphi = \langle \varphi_{xy} : (x, y) \in \mathcal{E} \rangle$$

satisfies the group frame conditions, by Frame Theorem 7.3 in [7], and \mathfrak{A} is essentially isomorphic to the full group relation algebra $\mathfrak{G}[\mathcal{F}]$, by (the proof of) Theorem 7.4 in [7]. The group relation algebra $\mathfrak{G}[\mathcal{F}]$ is one of the ones considered in Theorem 3.9, and therefore also in GCD Theorem 3.8 (up to isomorphism).

Alternatively, let $m_{xy} = |G_x/H_{xy}|$ and observe that the system

$$m = \langle m_{xy} : (x, y) \in \mathcal{E} \rangle$$

satisfies the index conditions by the proof of Scaffold Theorem 5.6. Consequently, \mathfrak{A} is essentially isomorphic to the cyclic group relation algebra constructed in GCD Theorem 3.8 using the system m . \square

Theorem 4.30 in Jónson-Tarski [9] states that for a relation algebra \mathfrak{A} the following are equivalent. (i) \mathfrak{A} is isomorphic to a full set relation algebra. (ii) \mathfrak{A} is complete, atomic, with all atoms x satisfying $x^\smile; 1; x \leq 1'$. The hard part of this theorem is to show that (ii) implies (i). Assume (ii) and let x be a subidentity atom. Then $x^\smile; 1; x \leq 1'$ by assumption, and thus $(x; 1; x)^\smile; (x; 1; x) = x^\smile; 1; x \leq 1'$ by Lemma 2.1(v) and Lemma 2.3(i),(ii). This means that $x; 1; x$ is functional. Thus the square $x; 1; x$ is the sum of one functional element, hence x is measurable with measure 1. Since \mathfrak{A} is atomic, the identity is a sum of atoms, and we have seen that these atoms are measurable, hence \mathfrak{A} is measurable. Each of the associated groups have one element, thus finite and cyclic. Since \mathfrak{A} is complete, then \mathfrak{A} is isomorphic, and not just essentially isomorphic, to a group relation algebra $\mathfrak{G}[\mathcal{F}]$ with all the groups in \mathcal{F} being one-element. It is not hard to see that such a $\mathfrak{G}[\mathcal{F}]$ is isomorphic to a full set relation algebra. We have proved the hard part of [9, Theorem 4.30] by using Theorem 5.7.

We note that a representation theorem is given in [2] which uses a generalization of the above condition (ii) in another direction, not toward measurability.

Next we turn to pair-dense relation algebras. Let \mathfrak{A} be a relation algebra. In [10], an element $x \in A$ is called a *pair* if $x; 0'; x; 0'; x \leq 1'$ and x is nonzero, where $0'$ denotes $-1'$, and the algebra \mathfrak{A} is called *pair-dense* if the identity element $1'$ is a sum of pairs.

Lemma 5.8. *Let \mathfrak{A} be an atomic relation algebra. Then (i) and (ii) below are equivalent.*

- (i) \mathfrak{A} is pair-dense.
- (ii) \mathfrak{A} is measurable with all the associated groups cyclic of order ≤ 2 .

Proof. In the proof, we will use Lemma 2.1-Lemma 2.3 without mentioning them. Let $x \leq 1'$ be an atom, in particular x is nonzero. First we show that x is a pair just in case it is measurable with measure ≤ 2 . Now, by definition, x is a pair just in case $x; 0'; x; 0'; x \leq 1'$, we are going to show that this latter holds just when $x; 0'; x$ is functional. Indeed, $x; 0'; x$ is functional, by definition, just

when $(x; 0'; x)^\sim; (x; 0'; x) \leq 1'$, but $(x; 0'; x)^\sim; (x; 0'; x) = (x; 0'; x); (x; 0'; x) = x; 0'; x; 0'; x$. Note that each subidentity element x is functional by $x^\sim; x = x \leq 1'$.

Assume now that x is a pair. Then $x; 1; x$ is the sum of $x; 1'; x$ and $x; 0'; x$, both being functional (since $x; 1'; x = x$). The first one, $x; 1'; x$ is nonzero by x being nonzero. If the second one, $x; 0'; x$ is zero, then x has measure 1, and if $x; 0'; x$ is nonzero, then x has measure 2. We have seen that x is measurable with measure ≤ 2 .

Assume now that x has measure ≤ 2 . Then x is the sum of ≤ 2 nonzero functional elements, and we mentioned at the beginning of Section 4 that each of these functional elements is an atom. Since x is a subidentity atom, it is nonzero and functional. Now, $x; 1; x = x; 1'; x + x; 0'; x$, where $x = x; 1'; x$. Thus, one of the functional elements below $x; 1; x$ is x itself. If x has measure 1, then $x; 1; x = x$ and so $x; 0'; x = 0$ hence functional. If x has measure 2, then the other functional element below $x; 1; x$ must be $x; 0'; x$. So in both cases $x; 0'; x$ is functional. We have already seen that $x; 0'; x$ is functional just in case x is a pair.

We are ready to prove the lemma. Assume that \mathfrak{A} is pair-dense and atomic. By definition and monotony, if x is a pair, then each nonzero element below it is also a pair. Therefore, $1'$ is the sum of pairs that are atoms. We have seen that all these atoms are measurable with measure ≤ 2 , hence (ii) holds. Assume now that (ii) holds, then $1'$ is the sum of measurable atoms with measure ≤ 2 . Each of these atoms is a pair, so \mathfrak{A} is pair-dense.

Finally, notice that each group of order ≤ 2 is cyclic. \square

In view of Lemma 5.8, our Theorem 5.7 implies that all atomic pair-dense relation algebras are completely representable, and in fact essentially isomorphic to a group relation algebra where the associated groups have order one or two. This gives a structural description for atomic pair-dense relation algebras. Theorem 48 of [10] states that simple pair-dense relation algebras are atomic, and Theorem 51 of [10] states that for simple pair-dense algebras (α) and (β) are equivalent, where (α) states that \mathfrak{A} is completely representable on the set U , and (β) states that the cardinality of U is $n + 2m$ where n is the number of atomic pairs x below $1'$ for which $x; 0'; x$ is zero, and m is the number for those where $x; 0'; x$ is nonzero. Now, using [10, Theorem 48] and Theorem 5.7, Lemma 5.8, one can give an alternative proof for [10, Theorem 51].

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